


Constructive proofs of the range property in lambda calculus

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 Dedicated in friendship to Corrado Böhm on the occasion of his 70th birthday

Abstract

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Böhm (1968) conjectured that the range of a combinator is either a singleton or an infinite set. The conjecture was proved independently by Myhill and the author. A proof is presented in Barendregt (1984) in a powerful – but somewhat difficult to understand – topological formulation due to Visser (1980). Dirk van Dalen remarked that the proof of the conjecture is not constructive. In this paper we first present some unsuccessful attempts to prove the conjecture, including the motivation given by Böhm. Then we present the proof as originally given by Barendregt and Myhill and we sketch the topological proof of Visser. After that we give two constructive proofs of the conjecture. The first one closely follows the original motivation by Böhm but has as an extra ingredient the notion of coding. The second proof is based on a recursion theoretic analysis of the situation in terms of Ershov numerations. Finally, we present some generalizations of the range theorem.

1. Böhm's conjecture

We use notations from [1]. In particular, for $F \in \Lambda^\circ$ let $\text{Ra}(F)$ be the set $\{FA \mid A \in \Lambda^\circ\}$ modulo β -convertibility, more precisely $\{[FA] \mid A \in \Lambda^\circ\}$, where $[M]$ is for $M \in \Lambda^\circ$ its equivalence class under the relation $=_\beta$ (we often write $=$ for $=_\beta$). The terms $\ulcorner n \urcorner$ denote the (Church) numerals $\lambda f x. f^n x$. For a closed term $M \in \Lambda^\circ$ we write $\# M \in \mathbb{N}$ (where \mathbb{N} is the set of natural numbers) for its code and $\ulcorner \# M \urcorner \equiv \ulcorner \# M \urcorner$ for the corresponding numeral. There exists a closed λ -term **E** that acts as an interpreter for closed terms: $\mathbf{E} \ulcorner M \urcorner =_\beta M$ for all $M \in \Lambda^\circ$; see, for example, [2] for a short construction.

In [5] the following theorem is proved.

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Theorem 1.1 (Böhm's theorem). *Let $M, N \in \Lambda^\circ$ have different $\beta\eta$ -nf's. Then there exists an $F \in \Lambda^\circ$ such that*

$$FM =_\beta \lambda xy.x,$$

$$FN =_\beta \lambda xy.y.$$

In that same classical paper the following corollary was proved.

Corollary 1.2. *Let Q_1, Q_2 be elements of $\text{Ra}(F)$ having different $\beta\eta$ -nf's. Then there exists a $Q \in \text{Ra}(F)$ differing from both Q_1 and Q_2 .*

Proof. Let $FP_i = Q_i$ for $i = 1, 2$. By Theorem 1.1 there exists a $G \in \Lambda^\circ$ such that

$$GQ_1 = P_2, \quad GQ_2 = P_1.$$

Consider $F \circ G$. By the fixed-point theorem there exists a Q such that

$$Q = F \circ G(Q).$$

Therefore, $Q \in \text{Ra}(F)$. We claim that $Q \neq_\beta Q_1, Q_2$. If, say, $Q =_\beta Q_1$, then

$$Q_1 = Q = F \circ G(Q) = F(G(Q_1)) = FP_2 = Q_2,$$

which contradicts the idea that Q_1, Q_2 have different $\beta\eta$ -nf's. \square

The corollary inspired Böhm to the following conjecture.

Conjecture 1.3. *Given any $F \in \Lambda^\circ$, $\text{Ra}(F)$ is either a singleton or an infinite set.*

2. Proof attempts

Attempt 2.1. Theorem 1.1 can be generalized to any finite set of different $\beta\eta$ -nf's; see [6] or [1, Corollary 10.4.14]. However, the proof of Corollary 1.2 does not carry over, since the term Q does not need to have a β -nf.

Notation. Let $M \in \Lambda^\circ$ and let x be a variable. We write $x \in_\beta M$ for

$$\forall N [N =_\beta M \Rightarrow x \in \text{FV}(N)].$$

Attempt 2.2. Let $F \in \Lambda^\circ$ be given and let x be a variable. We distinguish the following two cases.

Case 1: $x \notin_\beta Fx$. Then $Fx =_\beta N$ with $x \notin \text{FV}(N)$ for some N . But then

$$FA =_\beta N[x := A] \equiv N$$

for all A . Therefore, $\text{Ra}(F)$ is singleton $\{N\}$.

Case 2: $x \in_{\beta} Fx$. This means that the x in Fx cannot be converted out. Then one would expect that for different A , say for $A \in \{\Omega^{\lceil 0 \rceil}, \Omega^{\lceil 1 \rceil}, \Omega^{\lceil 2 \rceil}, \dots\}$, where $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$, the FA are also different. However, it is not clear how to prove this. Moreover, even this expectation is wrong; see the next result.

Proposition 2.3 (Plotkin [8]). *There exists a term $F \in \Lambda^{\circ}$ and a $Z \in \Lambda^{\circ}$ such that for all $X \in \Lambda^{\circ}$ one has $FX =_{\beta} Z$, but $Fx \neq_{\beta(\eta)} Z$ and therefore $x \in_{\beta} Fx$.*

Proof. See [1, exercises 17.3.26 and 17.3.27]; the term F to be constructed is called Ξ in this reference. \square

This proposition was motivated by the following corollary.

Corollary 2.4 (Omega incompleteness of the lambda calculus). *There exist terms $F, G \in \Lambda^{\circ}$ such that $\forall X \in \Lambda^{\circ} FX =_{\beta} GX$ but $Fx \neq_{\beta(\eta)} Gx$.*

Proof. Take F, Z as in the proposition and $G \equiv KZ$. \square

Note that Proposition 2.3 does not contradict Conjecture 1.3. Moreover, the method of the second proof attempt works well in showing the range property in continuous lambda models; see Section 5.

3. Proving the conjecture

The following proof of Böhm's conjecture has been given independently by Myhill and Barendregt. Remember that if a set $A \subseteq \mathbb{N}$ and its complement $\mathbb{N} - A$ are both r.e., then even A is recursive (this is sometimes called the “negation theorem”; see [9, Theorem II, p. 58] for a proof).

Proposition 3.1 (Scott's theorem). *If $A \subseteq \Lambda^{\circ}$ is a nontrivial (i.e. $A \neq \emptyset, A \neq \Lambda^{\circ}$) set closed under $=_{\beta}$ (i.e. $M \in A$ and $M =_{\beta} N \Rightarrow N \in A$), then A is not recursive (after coding).*

Proof. See [1, Theorem 6.6.2]. \square

Theorem 3.2 (Range property). *Let $F \in \Lambda^{\circ}$. Then $\text{Ra}(F)$ is either a singleton or an infinite set.*

Proof. Suppose that $\text{Ra}(F) = \{[M_1], [M_2], \dots, [M_k]\}$, with $k \geq 2$. Define $A_i = \{P \in \Lambda^{\circ} \mid FP =_{\beta} M_i\}$. Then each $A_i \subseteq \Lambda^{\circ}$ is (after coding) an r.e. set. Moreover, the complement of A_1 is $A_2 \cup \dots \cup A_k$ and is therefore also an r.e. set. Hence, by the negation theorem A_1 is recursive. But this contradicts Scott's theorem, since A_1 is a nontrivial set closed under $=_{\beta}$. \square

By the same proof one can show that the range property also holds modulo $\beta\eta$ -conversion and in fact modulo any r.e. theory T .

Dirk van Dalen remarked that, as the negation theorem is only valid using classical logic, the given proof of the range property is not constructive. In the next section we will nevertheless show that the range property holds constructively. Before doing that we will sketch the topological proof of the range property in [10].

Proof (sketch). Let $A \subseteq \Lambda^0$. Then A is called *Visser-open* if A is closed under $=_\beta$ and its complement $\Lambda^0 - A$ is r.e. The Visser open sets form a basis for the so-called *Visser topology*. The term model, which is Λ/\equiv_β , inherits the quotient topology. We have the following facts (see [10, 1]):

(i) Λ/\equiv_β with the Visser topology is a connected topological space. In fact, this space is *hyperconnected*, i.e. every two nonempty open sets have a nonempty intersection. This fact also follows immediately from Theorem 5.5.

(ii) Let $F \in \Lambda^0$. Then the map $[P] \mapsto [FP]$, i.e. F considered as a map on the term model, is continuous.

(iii) A finite subset of Λ/\equiv_β is a discrete subspace.

Now $\text{Ra}(F)$, as a continuous image of a connected space, is also connected. If $\text{Ra}(F)$ is finite, then it is discrete. But the only connected discrete space is a singleton. \square

4. Constructive proof of the range property

Theorem 4.1 (Range property, constructive version). *Let $F \in \Lambda^0$ be given. Assume that $\{P_1, \dots, P_k\} \subseteq \Lambda^0$, with $k \geq 2$, is a finite set such that*

$$\mathcal{A} = \{[FP_1], \dots, [FP_k]\}$$

consists of different elements of $\text{Ra}(F)$. Then one can effectively construct an element $[P] \in \text{Ra}(F) - \mathcal{A}$.

Proof. Write $Q_i \equiv FP_i$. There exists a partial recursive function χ such that for all $n \in \mathbb{N}$ one has

$$\chi(n) = \begin{cases} \# P_2 & \text{if } \mathbf{E}[n] =_\beta Q_1, \\ \# P_3 & \text{if } \mathbf{E}[n] =_\beta Q_2, \\ \vdots & \\ \# P_k & \text{if } \mathbf{E}[n] =_\beta Q_{k-1}, \\ \# P_1 & \text{if } \mathbf{E}[n] =_\beta Q_k, \\ \uparrow & \text{else.} \end{cases}$$

Let χ be λ -defined by $G \in \Lambda^0$. Consider $F \circ \mathbf{E} \circ G$. By the second fixed-point theorem there exists a $Q \in \Lambda^0$ such that

$$Q = F \circ \mathbf{E} \circ G[Q].$$

Then, clearly, $Q \in \text{Ra}(F)$. We show that $Q \notin A$. If $Q = Q_i$ for some $1 \leq i \leq k$, then

$$\chi(\#Q) = \#P_{i+1} \pmod k$$

since $\mathbf{E}^\top \#Q^\top = Q = Q_i$; hence, $G^\top Q^\top = \lceil P_{i+1} \rceil$. But then

$$Q_i = Q = F \circ \mathbf{E} \circ G^\top Q^\top = F \circ \mathbf{E}^\top P_{i+1} = FP_{i+1} = Q_{i+1},$$

contradicting the idea that the Q_j are different. \square

The result is uniform in the (code of the) finite set $\{P_1, \dots, P_k\}$.

It is interesting to note that this proof of the effective version of the range property is very similar to that of Corollary 1.2, which made Böhm formulate his conjecture. In fact, the use of Theorem 1.1 is too powerful. Rather than working with the terms, one should handle the codes of the terms. The second fixed-point theorem will then replace the first one.

It is remarkable that in order to prove the range property, it seems that one has to interpret λ -calculus within λ -calculus (by using notions like convertibility in order to define a partial recursive function that is later represented by a λ -term). Why did the more direct proof attempts not work?

Perhaps the reason is that the range property is really a result in recursion theory. The best formulation uses the notion of a numeration (sometimes called “numbered set”) of Ershov [7]; see [10, 3] for a short introduction. In particular, the precomplete numerations are of interest. See [9, Section 7.3] for the definition of the notion “creative”.

The notion of precomplete numeration comes from Ershov’s 1973 article [7]. He also formulated for these the fixed-point theorem (Theorem 4.6). Let \mathcal{R} be the set of unary recursive functions and \mathcal{PR} that of unary partial recursive functions.

Definition 4.2. (i) A *numeration* is a pair $\gamma = (v, S)$ with $v: \mathbb{N} \rightarrow S$ a surjection.

(ii) Given a numeration $\gamma = (v, S)$, define on \mathbb{N} the following equivalence relation:

$$n \sim_\gamma m \Leftrightarrow v(n) = v(m).$$

(iii) Let $\gamma_1 = (v_1, S_1)$ and $\gamma_2 = (v_2, S_2)$ be two numerations. A map $\mu: S_1 \rightarrow S_2$ is called a *morphism* from γ_1 to γ_2 , notation $\mu: \gamma_1 \rightarrow \gamma_2$, if for some $f \in \mathcal{R}$ one has $v_2 \circ f = \mu \circ v_1$ (see the diagram below).

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\quad f \in \mathcal{R} \quad} & \mathbb{N} \\ v_1 \downarrow & & \downarrow v_2 \\ S_1 & \xrightarrow{\quad \mu \quad} & S_2 \end{array}$$

The intuition behind a numeration $\gamma = (v, S)$ is that the elements of S are somewhat complicated, but have codes in \mathbb{N} . If $v(n) = s$, then n is called a code for s . Then $n \sim_\gamma m$ means that n and m code the same object of S . Moreover, $\mu: S_1 \rightarrow S_2$ is a morphism if “ μ can be computed by means of the codes”.

Example 4.3. (i) $\Lambda^0/_\beta = (E, \Lambda^0/_\beta)$ with $E(n) = [\mathbf{E}^\top n^\top]$ is a numeration.

(ii) $\mathbf{PR} = (\Phi, \mathcal{PR})$, with $\Phi(n) = \phi_n$, the n th partial recursive function, is a numeration.

Definition 4.4. Let γ be a numeration.

(1) γ is said to be *precomplete* if

$$\forall \psi \in \mathcal{PR} \exists f \in \mathcal{R} \forall n \in \mathbb{N} [\psi(n) \downarrow \Rightarrow f(n) \sim_\gamma \psi(n)].$$

Following [10], we say that f *totalizes* ψ *modulo* \sim_γ .

(2) γ is called *positive* if \sim_γ is an r.e. relation.

Proposition 4.5. (i) $\Lambda^0/_\beta$ is precomplete and positive.

(ii) \mathbf{PR} is precomplete.

Proof. (i) $\Lambda^0/_\beta$ is positive because $n \sim_E m$ iff $E^\top n^\top =_\beta E^\top m^\top$. Given $\psi \in \mathcal{PR}$, let $F \in \Lambda^0$ be a λ -defining term for ψ . Define $f(n) = \#(E \circ F^\top n^\top)$. Then if $\psi(n) \downarrow$ one has

$$\mathbf{E}^\top f(n)^\top =_\beta \mathbf{E}^\top E \circ F^\top n^\top =_\beta E \circ F^\top n^\top =_\beta E(F^\top n^\top) =_\beta \mathbf{E}^\top \psi(n)^\top;$$

hence $f(n) \sim_E \psi(n)$.

(ii) Given $\psi \in \mathcal{PR}$, define

$$\theta(n, m) = \phi_{\psi(n)}(m).$$

By the s - m - n theorem one has for some $f \in \mathcal{R}$ and all $n, m \in \mathbb{N}$

$$\theta(n, m) = \phi_{f(n)}(m).$$

Then $\psi(n) \downarrow \Rightarrow \phi_{f(n)} = \phi_{\psi(n)} \Rightarrow f(n) \sim_\gamma \psi(n)$ for all $n \in \mathbb{N}$ and we are done. \square

Theorem 4.6 (Fixed-point theorem). *Let $\gamma = (v, S)$ be a precomplete numeration. Then*

$$\forall \psi \in \mathcal{PR} \exists n \in \mathbb{N} [\psi(n) \downarrow \Rightarrow \psi(n) \sim_\gamma n].$$

Proof. Given $\psi \in \mathcal{PR}$, define $\chi(m) = \psi(\phi_m(m))$. Then $\chi \in \mathcal{PR}$, so there is an $h \in \mathcal{R}$ that totalizes χ modulo \sim_γ . Let $h = \phi_e$. Suppose $\psi(h(e)) \downarrow$. Then $\chi(e) = \psi(\phi_e(e)) = \psi(h(e)) \downarrow$. So $h(e) \sim_\gamma \chi(e) = \psi(h(e))$. Therefore, $n = h(e)$ satisfies our requirement. \square

Corollary 4.7. *Let $\gamma = (v, S)$ be precomplete. Let $\mu: S \rightarrow S$ be an endomorphism, i.e. $\mu: \gamma \rightarrow \gamma$. Then μ has a fixed point:*

$$\exists s \in S \mu(s) = s.$$

Proof. By the definition of morphism there is an $f \in \mathcal{R}$ such that $v \circ f = \mu \circ v$. By the theorem there is an $n \in \mathbb{N}$ such that $f(n) \sim_\gamma n$. Then $s = v(n)$ is a fixed point of μ :

$$\mu(s) = \mu(v(n)) = v(f(n)) = v(n) = s. \quad \square$$

Theorem 4.6 implies both the fixed-point theorem of λ -calculus and the recursion theorem; see [4].

Theorem 4.8 (Ershov [7]). *Let γ be a precomplete numeration. Let $A \subseteq \mathbb{N}$ be nontrivial (i.e. $A \neq \mathbb{N}$, $A \neq \emptyset$), r.e. and closed under \sim_γ . Then A is creative.*

Proof. Since A is r.e. we only have to show that $\mathbb{N} - A$ is productive. Let $W_e \subseteq \mathbb{N} - A$ be r.e. in order to construct a $c \notin A \cup W_e$. Since A is nontrivial, we can find $a, b \in \mathbb{N}$ such that $a \in A$, $b \notin A$. Then we have $a \not\sim b$ and $\forall w \in W_e \ w \not\sim a$. Define

$$\psi(x) = \begin{cases} b & \text{if } x \in A, \\ a & \text{if } x \in W_e \cup \{b\}, \\ \uparrow & \text{else.} \end{cases}$$

Then $\forall x [\psi(x) \downarrow \Rightarrow \psi(x) \not\sim x]$. By the fixed-point theorem for precomplete numerations, Theorem 4.6, there exists a c such that

$$\psi(c) \downarrow \Rightarrow \psi(c) \sim c.$$

Hence, $\psi(c) \uparrow$. But then $c \notin A \cup W_e$. The construction of c is effective in e . \square

Corollary 4.9 *Let $F, P_1, P_2 \in A^0$ be such that $FP_1 \neq_\beta FP_2$. Let $W \subseteq A^\circ$ be an r.e. set such that $FP_1 \notin_\beta W$. Then $FP_3 \notin_\beta \{FP_1\} \cup W$, for some P_3 .*

Proof. Consider

$$V = \{n \in \mathbb{N} \mid F(\mathbf{E}^\top n^\top) \in_\beta W\},$$

$$A = \{n \in \mathbb{N} \mid F(\mathbf{E}^\top n^\top) =_\beta FP_1\}.$$

Then A is nontrivial ($\neq P_1 \in A$ and $\neq P_2 \notin A$), r.e. and closed under \sim_E ; moreover, V is r.e. and satisfies $V \subseteq \mathbb{N} - A$. By the theorem there is an element $a_3 \notin A \cup V$. Now we can take $P_3 \equiv \mathbf{E}^\top a_3^\top$. \square

As an immediate corollary we obtain another proof of the constructive version of the range property. Indeed, if FP_1, \dots, FP_k are different, then we can apply Corollary 4.9 to FP_1, FP_2 and $W = [FP_2] \cup \dots \cup [FP_k]$. However, this proof is essentially the same as that for Theorem 4.1.

5. Generalizations

A better analysis of the essence of the range theorem was suggested to me by R. Statman.

Definition 5.1. Let $\gamma = (v, \mathcal{S})$ be a numeration.

- (1) $A \subseteq \mathbb{N}$ is called a *Visser set* w.r.t. γ if A is r.e. and closed under \sim_γ .
 (2) $\mathcal{A} \subseteq \mathcal{S}$ is called a *Visser set* w.r.t. γ if $v^{-1}(\mathcal{A})$ is r.e. (and hence automatically a Visser set).

Definition 5.2. (1) $\mathbf{V}(\gamma) = \{A \subseteq \mathbb{N} \mid A \text{ is a Visser set w.r.t. } \gamma\}$.

(2) $\mathcal{V}(\gamma) = \{\mathcal{A} \subseteq \mathcal{S} \mid \mathcal{A} \text{ is a Visser set w.r.t. } \gamma\}$.

It is easy to see that \mathbf{V} and \mathcal{V} are lattices under the operations \cup and \cap having a least largest element.

Definition 5.3. An element x of a lattice $\langle V, \leq, \cup, \cap \rangle$ is called *join-irreducible* (j.i.) if $y_1 \cup y_2 = x \Rightarrow y_1 \leq y_2$ or $y_2 \leq y_1$.

The following result is an immediate corollary of Visser's ADN theorem [10].

Proposition 5.4. Let $\gamma = (v, S)$ be a precomplete numeration. Let $B \subseteq \mathbb{N}$ be a nontrivial (i.e. $B \neq \mathbb{N}$) r.e. set closed under \sim_γ , i.e. such that

$$n \in B \text{ and } n \sim_\gamma m \Rightarrow m \in B.$$

Then $\forall \psi \in \mathcal{PR} \exists f \in \mathcal{R} \forall n \in \mathbb{N}$

$$\psi(n) \downarrow \Rightarrow f(n) \sim_\gamma \psi(n),$$

$$\psi(n) \uparrow \Rightarrow f(n) \notin B.$$

Proof. See [3]. \square

Following Visser we say that every $\psi \in \mathcal{PR}$ can be made total by $f \in \mathcal{R}$ modulo \sim_γ avoiding B .

Theorem 5.5 (Visser [10]). Let $\gamma = (v, \mathcal{S})$ be a precomplete numeration.

(1) \mathbb{N} is j.i. in $\mathbf{V}(\gamma)$.

(2) \mathcal{S} is j.i. in $\mathcal{V}(\gamma)$.

Proof. (1) By contradiction. For $A, B \in \mathbf{V}$ suppose that $A \cup B = \mathbb{N}$ and that there are $a \in A - B$, $b \in B - A$. Define

$$\psi(x) = \begin{cases} b & \text{if } x \in A, \\ \uparrow & \text{else.} \end{cases}$$

By Proposition 5.4 there is a recursive function f that makes ψ total modulo \sim_γ avoiding B . Hence, for all $x \in \mathbb{N}$ one has

$$\begin{aligned} x \in A &\Rightarrow \psi(x) \downarrow \\ &\Rightarrow f(x) \sim \psi(x) = b \\ &\Rightarrow f(x) \notin A, \end{aligned}$$

$$\begin{aligned}
x \notin A &\Rightarrow \psi(x) \uparrow \\
&\Rightarrow f(x) \notin B \\
&\Rightarrow f(x) \in A.
\end{aligned}$$

By the fixed-point theorem for precomplete numerations $f(n) \sim n$ for some n . But then A is not closed under \sim .

(2) Immediate by (1). \square

Corollary 5.6 (R. Statman). *Let $\gamma_1 = (v_1, \mathcal{S}_1)$ and $\gamma_2 = (v_2, \mathcal{S}_2)$ be two numerations and $\mu: \mathcal{S}_1 \rightarrow \mathcal{S}_2$ a morphism.*

(1) *Suppose $\gamma_1 = (v_1, \mathcal{S}_1)$ is precomplete. Then $\mu(\mathcal{S}_1)$ is j.i. in $\mathcal{V}(\gamma_1)$.*

(2) *Suppose γ_1 is precomplete and that γ_2 is positive. Then $\mu(\mathcal{S}_1)$ is either a singleton or an infinite set.*

Proof. (1) Suppose $\mu(\mathcal{S}_1) = \mathcal{X} \cup \mathcal{Y} \subseteq \mathcal{S}_2$ is a decomposition in r.e. sets. Then $\mathcal{X}' = \mu^{-1}(\mathcal{X})$, $\mathcal{Y}' = \mu^{-1}(\mathcal{Y})$ is such a decomposition of \mathcal{S}_1 . Therefore, by the theorem, say, $\mathcal{X}' \subseteq \mathcal{Y}'$. But then also $\mathcal{X} \subseteq \mathcal{Y}$.

(2) If $\mu(\mathcal{S}_1)$ is not a singleton, then it is the union of two sets; if moreover $\mu(\mathcal{S}_1)$ is finite, then it is the union of two r.e. sets, since each singleton is r.e. (γ_2 being positive). But this is impossible by (1). \square

Now $\Lambda^\circ / =_\beta$ is by Proposition 4.5 a positive and precomplete numeration. Since lambda terms induce morphisms on the term model seen as numeration, the range property for combinators follows as corollary. The following example from Statman shows that there is an endomorphism on the closed term model that is not induced by a combinator.

Consider the numeration $\Lambda^\circ / =_\beta$. Each combinator $F \in \Lambda^\circ$ induces an endomorphism $\mu_F: \Lambda^\circ / =_\beta \rightarrow \Lambda^\circ / =_\beta$ defined by $\mu_F([M]) = [FM]$. The following result shows that not every endomorphism is of the form μ_F .

Definition 5.7. Let $F, G \in \Lambda^\circ$ be such that $F \circ G = \mathbf{I}$. Then F, G determine an *inner model* of the lambda calculus as follows. Define the map $m = m_{F, G}: \Lambda^\circ \rightarrow \Lambda^\circ$ by

$$\begin{aligned}
m(x) &= x, \\
m(PQ) &= Fm(P)m(Q), \\
m(\lambda x.P) &= G(\lambda x.m(P)).
\end{aligned}$$

Lemma 5.8. *Given an inner model of the lambda calculus F, G , one has for $m = m_{F, G}$ for all $P, Q \in \Lambda$*

$$P =_\beta Q \Rightarrow m(P) =_\beta m(Q).$$

Proof. First show that $FV(P) = FV(m(P))$ by induction on the structure of $P \in \mathcal{A}$ and, moreover, for all $Q \in \mathcal{A}$

$$m(P[x := Q]) \equiv m(P)[x := m(Q)].$$

Then it follows that

$$m((\lambda x.P)Q) = m(P[x := Q]).$$

Now the required property can be proved by induction on the derivation of $P = Q$ in the lambda calculus. \square

Proposition 5.9 (R. Statman). *Every inner model F, G determines an endomorphism $\mu_{F,G}: \mathcal{A}^\circ / \equiv_\beta \rightarrow \mathcal{A}^\circ / \equiv_\beta$ defined by $\mu([P]) = [m(P)]$. Moreover for some F, G this endomorphism is not induced by a combinator.*

Proof. The first statement follows from the lemma. If $F \equiv \lambda x.x \mathbf{I}$ and $G \equiv \lambda xz.zx$, then F, G determine an inner model such that $\mu_{F,G}$ is not induced by a combinator. Indeed,

$$m_{F,G}((\lambda x.xx)(\lambda x.xx)) \equiv \langle \lambda x.x \mathbf{I} x \rangle \mathbf{I} \langle \lambda x.x \mathbf{I} x \rangle,$$

$$m_{F,G}(\lambda x.x) \equiv \langle \lambda x.x \rangle;$$

here $\langle P \rangle$ stands for $\lambda z.zP$. By some underlining technique (see [1, Ch. 17]) it can be shown that this cannot be accomplished by a morphism induced by a combinator; i.e. for no $H \in \mathcal{A}^\circ$ one has

$$H((\lambda x.xx)(\lambda x.xx)) = \langle \lambda x.x \mathbf{I} x \rangle \mathbf{I} \langle \lambda x.x \mathbf{I} x \rangle,$$

$$H(\lambda x.x) = \langle \lambda x.x \rangle. \quad \square$$

Therefore, Corollary 5.6 is a more general form of the range property.

The range property also holds in some models, like $P\omega$ and D_∞ . C. Wadsworth proved this by using the idea in the second proof attempt above; see [1], Theorem 20.2.6. Instead of the relation $x \in_\beta FV(Fx)$ one uses $x \in BT(Fx)$ and the so-called Böhm-out technique introduced in [5].

It is an open question whether the range property holds for the closed term model modulo the theory \mathcal{H} that identifies all unsolvable terms. We conjecture that it does; see [1, Exercises 20.6.9–20.6.11] for some evidence. We would like to encourage the reader to work on this conjecture. It is not clear whether the recursion theoretic method will work.

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Note added in proof

The referee has made the following observation.

Theorem. Suppose $\mathcal{A}_1, \dots, \mathcal{A}_n \subseteq \Lambda^\circ$, with $n > 1$, are disjoint Visser sets such that there are $P_i \in \mathcal{A}_i$. Then one can construct a term $Q \in \Lambda^\circ$ such that $Q \notin \mathcal{A}_1 \cup \dots \cup \mathcal{A}_n$.

Proof. Similar to the proof of Theorem 4.1. \square

In fact, Theorem 4.1 immediately follows as a corollary. On the other hand, this result also follows from Theorem 4.8.